Streamline jumping: A mixing mechanism

Ivan C. Christov,[1] Julio M. Ottino,[2,3,4] and Richard M. Lueptow[4,*]
[1]*Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois 60208, USA*
[2]*Department of Chemical and Biological Engineering, Northwestern University, Evanston, Illinois 60208, USA*
[3]*The Northwestern Institute on Complex Systems (NICO), Northwestern University, Evanston, Illinois 60208, USA*
[4]*Department of Mechanical Engineering, Northwestern University, Evanston, Illinois 60208, USA*
(Received 27 August 2009; published 12 April 2010)

We study a singular limit of tumbled granular flows in quasi-two-dimensional rotating drums, demonstrating that the limiting dynamical system, as the shear layer vanishes, belongs to a class of discrete discontinuous mappings called piecewise isometries. In doing so, we identify a mechanism of mixing, in the absence of the usual streamline crossing mediated by the flowing layer. By considering the exceptional case of a 50% full square tumbler, this mechanism (streamline jumping) is related to the horizontal motion of the free surface of the flow in non-half-full tumblers. The limiting dynamics are quite complex, if not (technically) chaotic.

DOI: 10.1103/PhysRevE.81.046307 PACS number(s): 47.52.+j, 45.70.Mg, 05.45.−a, 47.32.−y

I. INTRODUCTION

Chaotic advection, or *Lagrangian chaos*, is a paradigm [1] with applications ranging from geophysical transport [2] to the design of micromixers [3]. When applied to mixing processes, chaotic advection's essence lies in maximizing *streamline crossings* [1], an idea that has been formalized mathematically using *linked twist maps* [4].

Achieving chaotic mixing is fundamentally about modulating time-periodic flows, so that streamlines change in time and cross those at previous times, allowing particles to smoothly transfer from one streamline to another [1,3,4]. This approach has also been successfully applied to mixing in *tumbled granular flows* [5]. In the latter, streamline crossing occurs as a consequence of the changing length and thickness of the so-called flowing layer [5] or the time-varying rotation rate [6]. Here, we demonstrate that, in two dimensions, these flows can exhibit a related, but fundamentally different, behavior: particles can discontinuously jump between streamlines. This is because, as we discuss below, it is physically possible to make the region of streamline crossing vanishingly small.

To elucidate this, we focus purely on the kinematics of motion in tumbled granular flows, i.e., the motion of particles in a convex (for our purposes) quasi-two-dimensional (quasi-2D) (meaning, the thickness is negligible compared to the height and width) rotating container. Studies have shown that the mixing and segregation of particulate matter are predominantly controlled by the geometry of the container [5,6]. Therefore, a kinematic description of the flow suffices.

The granular matter in the container, a square one being illustrated in Fig. 1, is assumed to fill a certain fraction ϕ of the available volume and to be in the regime where particles are continuously flowing down the free surface, forming a thin lens-shaped shear layer (the *flowing* or *fluidized layer*), while the rest of the particles (the *bulk* or *fixed bed*) are in solid-body rotation, about the point C, with the container [7–9]. Also, we assume that the flow is not fast enough to deform the free surface to the point where it is no longer flat.

We are interested in the behavior and the predictions of the mathematical model as the fluidized layer becomes very thin in the example square tumbler (although the results carry over to all convex shapes). Recently, this was briefly noted in a three-dimensional spherical tumbler [10], and an unexpected connection was found to *piecewise isometries* (PWIs) [11,12], a branch of dynamical systems theory. Much like how linked twist maps are the fundamental underlying feature of "good" fluid mixing [4], it now appears that piecewise isometries may play a similar role in granular mixing. Even though these types of mappings do not possess the stretching property characteristic of chaotic flows [10,11], we show that they can nevertheless lead to seemingly chaotic mixing of particulate matter in a tumbler.

II. CONTINUUM MODEL OF TUMBLED GRANULAR FLOWS

Although a number of continuum models are available, depending on the granular phenomenon being studied

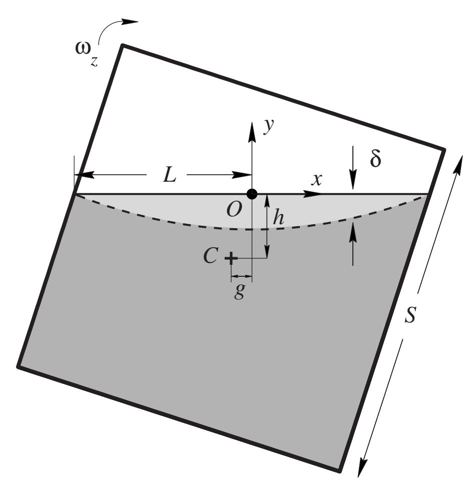

FIG. 1. Diagram of a 70% full square quasi-two-dimensional tumbler, which has been rotated backward by the dynamic angle of repose (so that the free surface of the flow is horizontal), showing the coordinate system and notation. The thickness of the tumbler is assumed much smaller than its width and height, so it is not shown. The boundary of the flowing layer is represented by a dashed curve.

*r-lueptow@northwestern.edu

[13–17], the simplest one for granular mixing is the kinematic model [5,6]. Following the standard approach to chaotic advection [1,18], if we let $(x(t),y(t))$ be a pathline in this tumbled flow, under which the motion is piecewise defined (as shown in Fig. 1, with $\omega_z>0$), then the dynamic equations of its evolution in the moving frame take the form

$$\frac{d}{dt}x(t)=\begin{cases} v_x(x(t),y(t),t), & y(t)>-\delta(x(t),t),\\ \omega_z[y(t)+h(t)]-\dot{g}(t), & \text{otherwise},\end{cases} \tag{1a}$$

$$\frac{d}{dt}y(t)=\begin{cases} v_y(x(t),y(t),t), & y(t)>-\delta(x(t),t),\\ -\omega_z[x(t)+g(t)]-\dot{h}(t), & \text{otherwise}.\end{cases} \tag{1b}$$

Within the flowing layer, under the assumption that the depth-averaged streamwise velocity is independent of the streamwise position [5], the Eulerian velocity field is

$$v_x(x,y,t)=2\bar{v}_x(t)[1+y/\delta(x,t)], \tag{2a}$$

$$v_y(x,y,t)=-\omega_z x[y/\delta(x,t)]^2, \tag{2b}$$

where the shape of the flowing layer and the depth-averaged streamwise velocity in it, respectively, are

$$\delta(x,t)=\delta_0(t)\left\{1-\left[\frac{x}{L(t)}\right]^2\right\},\quad \bar{v}_x(t)=\frac{\omega_z L(t)^2}{2\delta_0(t)}. \tag{3}$$

Note that this model presumes that the streamwise velocity in the flowing layer depends linearly on the depth, as a first approximation to the experimentally obtained velocity profile [19–21]. Other flowing layer shapes and velocity profiles can be derived under different assumptions [6,21,26,27]. Without loss of generality, we consider only the original model [5], since mixing under all kinematic models is qualitatively the same [6].

The maximal depth of the flowing layer $\delta_0(t)$ and the half-length of the free surface $L(t)$ are known functions of time alone and are such that $\varepsilon := \delta_0(t)/L(t)$ can be assumed to be a constant *independent* of time [5]. This experimentally motivated assumption is termed the *geometric similarity* of the flowing layer because, physically, it means that the flowing layer adjusts instantaneously to changes in the container's orientation. The model has only one free parameter, namely, ε, which is fitted based on experimental data [5,6,19]. Once the geometry is specified, the fill fraction ϕ can also be varied.

For fixed t and $\varepsilon>0$, the velocity field (2) has streamlines shown for two overlapping tumbler configurations in Fig. 2. For any tumbler geometry, the change from a short thin flowing layer in orientation A to a long thick one in orientation B results in streamline crossing and hence chaotic advection [5].

III. VANISHING FLOWING LAYER LIMIT

Experiments [19] have shown that δ_0 is typically 5–12 particle diameters, which is much less than the length of the free surface $2L$, so $\varepsilon \ll 1$, meaning the limit $\varepsilon\to 0$ is a physically relevant one. The most important consequence of letting $\varepsilon\to 0$ is that $\delta_0\to 0$ also because $\varepsilon\equiv\delta_0/L$ and L is finite. Thus, the flowing layer becomes an infinitely thin *interface* as ε vanishes, collapsing onto the free surface of the granular flow.

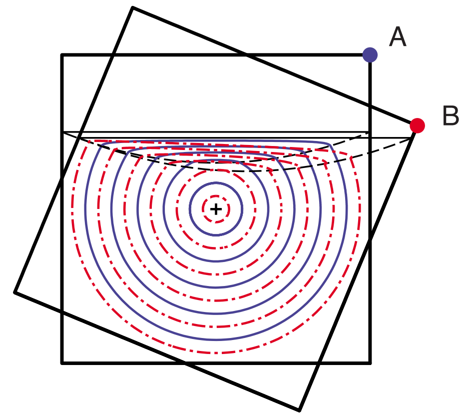

FIG. 2. (Color online) To-scale illustration of streamline crossing in a 75% full square tumbler with $\varepsilon=0.2$. Blue (solid) and red (dashed-dotted) curves are streamlines of the flow in tumbler orientations A and B, respectively. Note that they cross in the region where the flowing layers of the two orientations overlap, but they do not cross in the bulk. Orientation B is orientation A rotated 22.5° clockwise.

Also, we can evaluate the mean time-averaged speed of a particle over the length of the free surface,

$$\bar{V}_{\text{surf}}\equiv\frac{1}{T_f}\int_0^{T_f}\frac{1}{2L}\int_{-L}^{+L}\sqrt{v_x^2+v_y^2}\bigg|_{y=0}dx\,dt=\frac{\omega_z\bar{L}}{\varepsilon}, \tag{4}$$

where $\bar{L}=(1/T_f)\int_0^{T_f}L(t)dt$, $T_f=(2\pi/\omega_z)/n$ is the flow period, and n is the number of sides of the tumbler. Consequently, $\bar{V}_{\text{surf}}\to\infty$ as $\varepsilon\to 0$, since $\omega_z\bar{L}$ is a finite constant. Then, from Eq. (2a), the streamwise simple shear rate is

$$\dot{\gamma}(x,t)\equiv\frac{\partial v_x}{\partial y}=\frac{2\bar{v}_x(t)}{\delta(x,t)}=\frac{\omega_z L(t)^2}{\delta(x,t)\delta_0(t)}=\frac{1}{\varepsilon^2}\frac{\omega_z\delta_0(t)}{\delta(x,t)}. \tag{5}$$

Since $\omega_z\delta_0(t)/\delta(x,t)$ is bounded away from zero for all x and t, $\dot{\gamma}\to\infty$ as $\varepsilon\to 0$. Therefore, in the limit, the flowing layer becomes an "infinitely strong" shear.

Finally, we assume that when $\varepsilon=0$ particles do indeed traverse the flowing layer *instantaneously* ($\bar{V}_{\text{surf}}=\infty$), but they leave it at the streamwise position (on the interface) corresponding to the reflection of the position they entered across the free surface midpoint O (Fig. 1), as is the case when $\varepsilon>0$ and $\delta_0=$const. That is, if a particle reaches $y=0$ at $x=x_{\text{enter}}(<0)$, based on the coordinate system with the origin at O, at time $t=t_{\text{enter}}$, then it is instantaneously transferred to $x=-x_{\text{enter}}$. Using the rigid coordinate system with the origin at C, $\tilde{x}=-\tilde{x}_{\text{enter}}+2g(t_{\text{enter}})$ is the new location of the particle, where $g(t)$ is the time-dependent horizontal displacement of O with respect to C. If $g(t_{\text{enter}})\neq 0$, then $\tilde{x}\neq-\tilde{x}_{\text{enter}}$, and the particle will not remain on the same streamline after the jump.

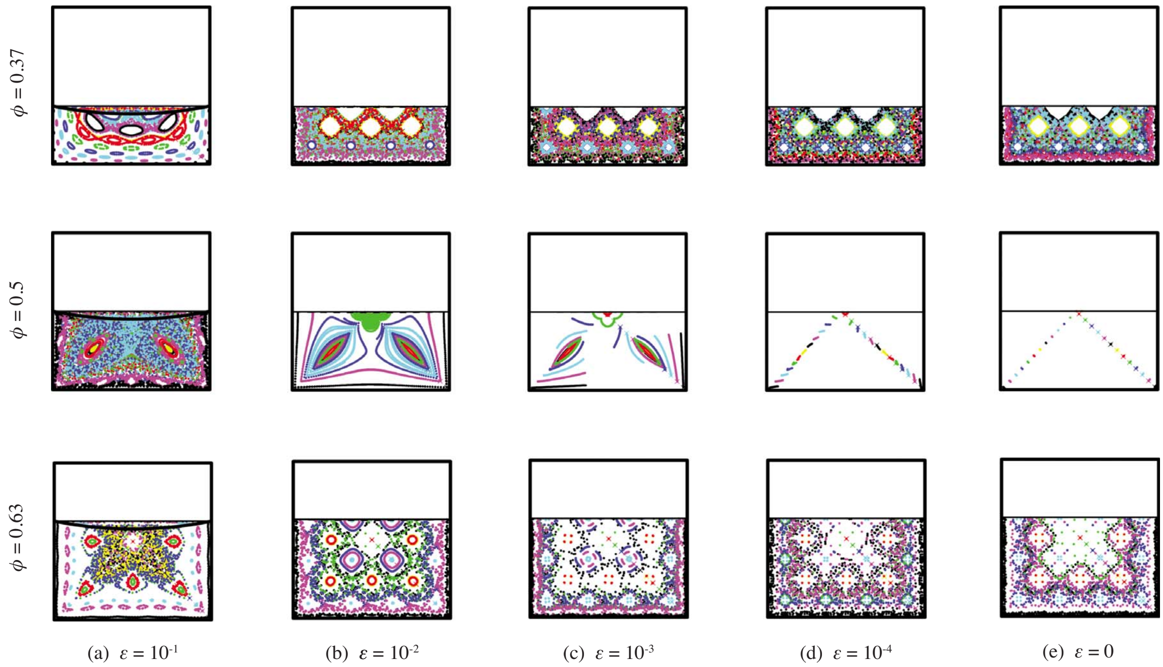

FIG. 3. (Color online) 500-period Poincaré sections for three fill fractions of a square tumbler (37% on the top row, 50% on the middle row, and 63% on the bottom row) and various values of ε, starting with $\varepsilon = 0.1$ (a situation easily realized experimentally).

IV. OVERVIEW OF THE DYNAMICS

The mixing and segregation of bidisperse granular matter can be related to the Poincaré section of the kinematic model of the flow [6]. To this end, in Fig. 3, sample numerical Poincaré sections [22] are shown for various ε's and ϕ's in the example square tumbler geometry.

Although many transitions and bifurcations can occur in the Poincaré section's pattern as ε and ϕ are varied, the basic trend is that at $\varepsilon \approx 0.01$, for any ϕ, the Poincaré section has qualitatively reached a consistent pattern (i.e., the number of elliptic "islands" and their location remains the same for all $\varepsilon \lesssim 0.01$). However, $\phi = 0.5$ is an *exception* in that the chaotic region is lost as the flowing layer vanishes, with the islands dominating the Poincaré section for finite ε. Therefore, it is not only the container's noncircular shape that induces time-periodic disturbances leading to chaos when $\varepsilon = 0$, as conjectured in [23] where this case was studied analytically, but also the interaction of the fill fraction with the tumbler shape.

V. STREAMLINE JUMPING: THE MECHANISM OF MIXING FOR $\varepsilon = 0$

In the tumbled granular flow, streamlines in the bulk, where all particles undergo solid-body rotation, are always arcs of concentric circles; no streamline crossing can occur there. In the flowing layer of a noncircular tumbler, however, the changing maximal depth $\delta_0(t)$ modulates the streamlines in time creating crossings (Fig. 2). Thus, it may seem that if the flowing layer vanishes ($\varepsilon \to 0$), then there cannot be chaotic mixing since streamline crossing is impossible. However, the chaotic regions present for $\phi = 0.37$ and 0.63 in Fig. 3 clearly show that this is false. Thus, good mixing can be achieved despite the apparent lack of good conditions for mixing. But what is the underlying ϕ-dependent physical mixing mechanism?

In a noncircular container, the free surface moves in time in a manner parametrically dependent on the fill fraction [6]. More precisely, the midpoint of the flowing layer O (the point across which a particle is reflected when $\varepsilon = 0$) moves horizontally according to $\tilde{x}_O = g(t)$. Hence, upon traversing the infinitely thin flowing layer, particles do *not necessarily* emerge onto another portion of the same streamline; they can "jump" between different solid-body-rotation streamlines.

This idea is illustrated in Fig. 4 for the example square tumbler. When $\phi = 0.27$ [Fig. 4(a)], a particle initially on the solid blue streamline, when the tumbler is in orientation A, can jump onto the dashed red streamline upon reaching the flowing layer at some $t > 0$ (illustrated by the horizontal cyan arrow), when the tumbler is in orientation B. The flowing

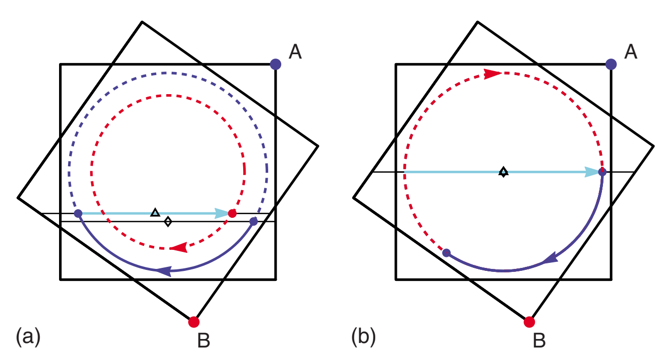

FIG. 4. (Color online) To-scale illustration of how streamline jumping can occur in two tumblers with infinitely thin flowing layers ($\varepsilon = 0$) but with different fill fractions: (a) $\phi = 0.27$ and (b) $\phi = 0.5$. Note the absence of a streamline jumping mechanism in (b). In both, the rotation is clockwise.

layer's midpoints in orientations A and B are denoted by a diamond and a triangle, respectively; notice that these do not coincide for $\phi=0.27$. In Fig. 4(b), where $\phi=0.5$ now, the midpoint of the flowing layer remains coincident with the center of rotation of the tumbler for all times, i.e., $g(t)=h(t)\equiv 0\,\forall\, t$. Clearly, no streamline jumping is possible, and particles are always transferred to the same streamline. Hence, complex dynamics are not possible in a 50% full even-sided polygonal tumbler when $\varepsilon=0$, confirming the numerical results shown in Fig. 3.

This geometric reasoning is valid for *any* noncircular tumbler, giving a *necessary condition* for complicated particle trajectories. Namely, if the displacements of the moving coordinate system $g(t)$, $h(t)$ are *not* identically zero at a given ϕ, then there exists a mechanism of mixing, unrelated to diffusion (i.e., particle dispersal through random collisions), despite the absence of a flowing layer and streamline crossing.

To understand how this mechanism mixes, imagine a line segment entering the infinitely thin flowing layer parallel to it. The entire line segment is reflected across O instantaneously, and its length remains unchanged. However, this will almost never be the case. If the segment enters the flowing layer at an angle, each point on is reflected across O at a slightly different time and tumbler orientation. This shifts the points on the segment to nearby streamlines, eventually spreading them further apart than they were initially, which results in mixing.

VI. CONNECTION TO PIECEWISE ISOMETRIES

When $\varepsilon=0$, particles in the flow will always undergo rotations (a type of distance-preserving mapping or isometry), occasionally crossing a line of discontinuity (the flowing layer) that transfers them to another trajectory of solid-body rotation. According to [12], this qualifies the limiting dynamical system as a PWI. PWIs can exhibit the usual behaviors of nonlinear dynamical systems: periodic points, fractal structures, global attractors, and generally complex dynamics [10–12].

However, one crucial difference exists between a PWI exhibiting complex behavior and a dynamical system exhibiting chaotic behavior: in the case of the former the divergence of nearby trajectories is conjectured to be at most *algebraic*, while in the latter it is *exponential* [10], asymptotically as $t\to\infty$. Mathematically, this is because (under certain conditions) PWIs have *zero topological entropy* [24], meaning they do not possess the stretching characteristics that are universal of chaotic dynamical systems [4] and fluid mixing [1,25]. Therefore, even though on a long enough time scale PWI-type dynamics lead to inherently slower mixing than the usual chaotic dynamics, for short times the opposite can be true.

This can be observed in Figs. 3(b)–3(d). In the 50% full tumbler, the stretching and folding mechanism is quite weak (due to the limited streamline crossings, which disappear as $\varepsilon\to 0$), as the lack of chaos in these Poincaré sections shows. Meanwhile, for the 37% and 63% full tumblers, where streamline jumping (i.e., the PWI-type mechanism) is possible, the Poincaré sections have large chaotic regions, showing that the PWI-type mixing mechanism dominates here [28].

VII. CLOSURE

In the limiting dynamical system governing tumbled granular flows streamline crossing is impossible, yet chaotic advection (in a sense yet to be made mathematically precise) persists due to streamline jumping. This mechanism leads to complex dynamics in a physical flow without stretching and folding. Without parallel in the mixing of fluids, the uniquely granular phenomenon of streamline jumping leads (counterintuitively) to good mixing even when the flowing layer is vanishingly thin. The connection between the kinematics of granular flow and the mathematics of PWIs discovered here for quasi-2D tumblers, along with the preliminary results in [10], shows that these types of discrete discontinuous dynamical systems are a generic and key underlying mechanism in granular mixing.

ACKNOWLEDGMENTS

We thank Stephen Wiggins and Rob Sturman for their careful reading of the manuscript and their valuable input.

[1] J. M. Ottino, Annu. Rev. Fluid Mech. **22**, 207 (1990).
[2] R. Pierrehumbert and H. Yang, J. Atmos. Sci. **50**, 2462 (1993).
[3] J. -M. Hertzsch, R. Sturman, and S. Wiggins, Small **3**, 202 (2007).
[4] R. Sturman, J. M. Ottino, and S. Wiggins, *The Mathematical Foundations of Mixing* (Cambridge University Press, Cambridge, England, 2006).
[5] K. M. Hill, D. V. Khakhar, J. F. Gilchrist, J. J. McCarthy, and J. M. Ottino, Proc. Natl. Acad. Sci. U.S.A. **96**, 11701 (1999).
[6] S. W. Meier, R. M. Lueptow, and J. M. Ottino, Adv. Phys. **56**, 757 (2007).
[7] G. Metcalfe, L. Graham, J. Zhou, and K. Liffman, Chaos **9**, 581 (1999).
[8] J. Rajchenbach, Phys. Rev. Lett. **65**, 2221 (1990).
[9] G. H. Ristow, *Pattern Formation in Granular Materials* (Springer-Verlag, Berlin, 2000).
[10] R. Sturman, S. W. Meier, J. M. Ottino, and S. Wiggins, J. Fluid Mech. **602**, 129 (2008).
[11] A. Goetz, in *Fractals in Graz 2001*, edited by P. Grabner and W. Woess (Birkhäuser, Basel, 2002), pp. 135–144.
[12] J. H. B. Deane, Meccanica **41**, 241 (2006).
[13] I. S. Aranson and L. S. Tsimring, Phys. Rev. Lett. **82**, 4643 (1999).
[14] S. P. Pudasaini and K. Hutter, *Avalanche Dynamics* (Springer-Verlag, Berlin, 2007).

[15] Z. S. Khan and S. W. Morris, Phys. Rev. Lett. **94**, 048002 (2005).

[16] J. M. N. T. Gray, J. Fluid Mech. **441**, 1 (2001).

[17] I. Goldhirsch, Annu. Rev. Fluid Mech. **35**, 267 (2003).

[18] H. Aref, Philos. Trans. R. Soc. London, Ser. A **333**, 273 (1990).

[19] N. Jain, J. M. Ottino, and R. M. Lueptow, J. Fluid Mech. **508**, 23 (2004).

[20] H. Chou and C. Lee, Granular Matter **11**, 13 (2009).

[21] A. V. Orpe and D. V. Khakhar, Phys. Rev. E **64**, 031302 (2001).

[22] In all Poincaré sections, 13 tracer particles are advected numerically for 500 periods. The angular speed of the tumbler is $\omega_z = 2\pi$ in the clockwise direction and its side length is $S = \sqrt{2}$.

[23] T. Elperin and A. Vikhansky, Chaos **9**, 910 (1999).

[24] J. Buzzi, Ergod. Theory Dyn. Syst. **21**, 1371 (2001).

[25] G. A. Voth, G. Haller, and J. P. Gollub, Phys. Rev. Lett. **88**, 254501 (2002).

[26] T. Elperin and A. Vikhansky, EPL **42**, 619 (1998).

[27] H. A. Makse, Phys. Rev. Lett. **83**, 3186 (1999).

[28] Strictly speaking, streamline jumping and PWI dynamics are only present when $\varepsilon \equiv 0$; however, for $0 < \varepsilon \ll 1$, $\bar{V}_{\mathrm{surf}} \gg 1$, and $\delta_0(t) \ll 1$, and (as in perturbation theory) the $\varepsilon = 0$ behavior is the "skeleton" of the dynamics close to this state (Fig. 3). Also, in this context, 500 periods can be considered a "short" time.