Diffraction and attenuation of a tone burst in mono-relaxing media

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(Received 6 August 2002; revised 20 June 2003; accepted 30 June 2003)

The influence of intrinsic absorption in a relaxing medium and the resulting three-dimensional diffraction correction of the magnitude of the acoustic pressure averaged over the surface of a receiver is investigated for a tone burst. A rigorous formula for the damped acoustic pressure average at the receiver was obtained for arbitrary pulsed waves in a mono-relaxing medium. Depending on the pulse oscillation frequency, envelope duration, and relaxation frequency of the media, the plane wave burst envelope can be reduced, amplified, or otherwise deformed. © *2003 Acoustical Society of America.* [DOI: 10.1121/1.1602701]

PACS numbers: 43.35.Ae, 43.35.Fj, 43.20.El [JGH]

I. INTRODUCTION

The prediction of diffraction for a transient burst propagating through an acoustic medium that exhibits relaxation is of interest for interpreting measurements of acoustic attenuation. In this paper, we develop a theory for the three-dimensional diffraction of a tone burst signal in a relaxing medium. Previous research has addressed separately various pieces of the problem such as attenuation due to three-dimensional diffraction of a harmonic wave, attenuation due to relaxational losses of a plane harmonic wave, or attenuation of a single plane pulse due to relaxational losses. However, to our knowledge no previous study has addressed the combined problem of attenuation due to diffraction (for a three-dimensional wave rather than for plane wave) for a burst signal (as opposed to a harmonic wave or a single pulse), while including attenuation due to relaxational losses (as opposed to ignoring relaxational losses). It is this problem that is addressed here.

We assume that a rigid circular planar piston with a uniform distribution of normal velocity on its surface is embedded in a rigid infinite baffle and radiates into the half-space of an ideal homogeneous, isotropic, relaxing medium. An acoustically transparent receiver with uniform sensitivity over its surface and a uniform amplitude-frequency response is coaxial with the piston. We are interested in the rigorous calculation of the average pressure exerted on the receiver by the three-dimensional acoustic pressure burst after propagating through a relaxing medium. Although a comprehensive investigation of this problem has not been reported, certain aspects of the problem have been considered by previous researchers.

Several methods to correct for the attenuation of a three-dimensional harmonic steady-state pressure wave due to diffraction (but not attenuation due to relaxational losses) have been described based on various models of the harmonically excited field in a loss-less medium.[1] Seki *et al.*[2] used approximate formulas for the average pressure over the piston in the absence of absorption based on the Lommel[3] and Williams[4] harmonic wave solutions of the Helmholtz equation in an attempt to account for diffraction effects in the measurement of acoustic attenuation. Khimunin[5,6] also applied the loss-less formula of Williams[4] for the average harmonic pressure to the diffraction correction of attenuation, demonstrating the influence of intrinsic absorption on the harmonic average pressure of the piston. Rogers and VanBuren[7] obtained a simple expression for the diffraction correction using Lommel's expression, which is a high-frequency limit of Williams' exact expression.

In addition to attenuation due to diffraction, classical attenuation and relaxational attenuation reduce the amplitude of a harmonic acoustic wave as it propagates. The theory of harmonic wave attenuation in relaxing medium was formulated by Kneser[8] and Mandel'shtam and Leontovich.[9] There is a substantial experimental and theoretical literature addressing relaxation in gases. We have recently extended the theoretical prediction of relaxational attenuation for a harmonic wave to a three-component gas mixture.[10]

The evolution of a single plane pulse (rather than a harmonic wave) in the absence of diffraction or scattering in relaxing medium has also been studied. A thorough analysis of the propagation of plane pulses in a relaxing media with relaxation laws modeling a variety of homogeneous materials was made in the review of Vainshtein,[11] with a focus on electromagnetic waves. Detailed analyses of the propagation of a single plane-wave acoustic pulse in the case of a mono-relaxing medium were performed analytically and numerically by Dunin,[12] Dunin and Maksimov,[13] Andreev *et al.*,[14] Larichev and Maksimov,[15] and Andreev *et al.*[16] Their results show a broadening of the pulse and a decrease in the pulse amplitude as the pulse propagates in space. In some cases, the theory was shown to match experiments reasonably well,[14,16] although in other cases differences between the measurement and theory were attributed to diffraction ef-

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fects, which are not taken into account when assuming a plane wave.[16]

The aim of this paper is to calculate analytically the diffraction correction to the attenuation of a three-dimensional tone burst in a mono-relaxing gas medium. To put this into context, the three-dimensional effects of diffraction on a harmonic wave in a lossless medium have been studied, as have the effects of relaxational losses on a harmonic wave or a single pulse for a one-dimensional plane wave. However, in order to recover the attenuation of sound from practical experiments in a relaxing medium, it is essential to formulate a model that can describe both the attenuation due to relaxational acoustical properties of the relaxing medium and the attenuation due to the three-dimensional distortion of acoustic wave. In addition, since many experiments use a short burst of sound to avoid reflections in the measured quantities, neither the harmonic wave nor the single pulse analysis is appropriate. Thus, the aim of this paper is to calculate analytically the diffraction correction to the attenuation of a three-dimensional tone burst in a mono-relaxing gas medium.

The solution will be based on the application of a Green's function to the three-dimensional wave equation for acoustical pressure in a medium with relaxation and averaging the pressure over the surface of the receiver. Consequently, the dependence of total attenuation of the average acoustic pressure on diffraction and relaxational attenuation can be separated. A result is that the phenomenon of distortion of the complex envelope of the oscillating pulse as it propagates can be represented by a relation that depends on the relaxation time of medium and the frequency of pulse oscillation.

II. THEORETICAL TREATMENT

A linear acoustic wave equation for acoustic pressure in a mono-relaxing gas medium has the form suggested by Rudenko and Soluyan,[17]

$$\frac{\partial^2 p}{\partial t^2} - c_0^2 \Delta p - m c_0^2 \Delta \int_{-\infty}^{t} \frac{\partial p}{\partial t'} e^{-(t-t')/\tau_{\text{relax}}} \partial t' = 0, \qquad (1)$$

where τ_{relax} is the relaxation time for the relaxation process, $m = (c_\infty^2 - c_0^2)/c_0^2$ is the net increase in phase speed as frequency varies from zero to infinity which characterizes the relaxation strength, c_0 is the equilibrium speed of sound, and c_∞ is the frozen speed of sound. Typically, parameter m satisfies the condition $m \ll 1$.

We assume a transient source condition for normalized pressure,

$$p(r,0,t) = H(a-r) M(t) e^{i\omega_0 t}, \qquad (2)$$

where H is the Heaviside function, r is radial coordinate, a is the radius of the piston, and the burst envelope $M(t)$ is a slowly varying function of time in comparison with the period of the oscillating component, $e^{i\omega_0 t}$.

The average pressure on an acoustically transparent receiver placed coaxial in relation to the piston source depends on the spacing z between transducer and receiver so that

$$\langle p(z,t) \rangle = \frac{1}{S} \int p(r,z,t)\, dS, \qquad (3)$$

where S is the area of receiver and angled brackets denote an averaged value.

We suppose for brevity that the radius of the receiver is equal to that of the transducer. In order to calculate the average pressure of Eq. (3), the Fourier transform in the time domain is applied to Eq. (1) using boundary condition Eq. (2). Then the average pressure can be represented for all frequencies in the form of Fourier integral, where use was made of the formula of average harmonic pressure of Williams[4]

$$\begin{aligned} \langle p(z,t) \rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \langle p(z,\omega) \rangle e^{i\omega t}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Bigg[e^{-ik(\omega)z} \\ &\quad - \frac{4}{\pi} \int_0^{\pi/2} e^{-ik(\omega)(z^2 + 4a^2 \cos^2\theta)^{1/2}} \sin^2\theta\, d\theta \Bigg] \\ &\quad \times \overline{M}(\omega - \omega_0) e^{i\omega t}\, d\omega. \end{aligned} \qquad (4)$$

The expression in the brackets is identical to that derived by Williams[4] for a harmonically oscillating piston in a lossless medium, where the wave number is real. In Eq. (4), the wave number $k(\omega)$, which is the dispersion relation of relaxing medium, is a complex function of frequency. The first term in the brackets is that for a plane harmonic wave; the second term in the brackets is the diffraction correction for a plane acoustic wave.

If the envelope function in Eq. (2) is set to unity, $M(t) = 1$, the piston source becomes a harmonic oscillation. In such a case the Fourier transform of $M(t)$ is the Dirac delta-function, and the formula for the average oscillating pulse pressure (4) simplifies to the formula of the average harmonic pressure of Williams [his Eq. (16)],[4] which is based on Eq. (1) with the relaxation terms omitted. Therefore, Eq. (4) for the average pressure of a burst in a mono-relaxing medium is a generalization of the Williams formula (which is limited to a harmonic wave in a lossless medium).

The dispersion relation for Eq. (1) can be approximated following Vainshtein[11] and Andreev *et al.*[14] by the expression

$$k(\omega) = \frac{\omega}{c_0} \left(1 + \frac{m\omega\tau_{\text{relax}}}{1 + i\omega\tau_{\text{relax}}} \right)^{1/2} \approx \frac{\omega}{c_\infty} + \frac{m'\omega}{2c_\infty(1 + i\omega\tau_{\text{relax}})} \qquad (5)$$

with an accuracy of m'^2, where $m' = 2(c_\infty - c_0)/c_0$. The two forms for the net increase in phase speed, m and m', are of the same order of magnitude because $m - m' = (c_\infty - c_0)^2/c_0^2 \ll 1$.

The dispersion relation Eq. (5) can be used to transform the average pressure Eq. (4) into a form suitable for the separate investigation of the attenuation of the oscillating portion of the burst and the envelope of the burst for the source Eq. (2). First, let

$$\xi = \omega - \omega_0 \qquad (6)$$

so that the Fourier integral Eq. (4) becomes

$$\langle p(z,t)\rangle \approx \frac{e^{i\omega_0 t}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[e^{-ik(\xi+\omega_0)z} - \frac{4}{\pi}\int_0^{\pi/2} e^{-ik(\xi+\omega_0)(z^2+4a^2\cos^2\theta)^{1/2}}\sin^2\theta\, d\theta\right] \times \overline{M}(\xi)e^{i\xi t}\, d\xi. \tag{7}$$

The form of the dispersion relation Eq. (5) makes possible the algebraic transformation in Eq. (7), leading to a modified dispersion relation of the form

$$k(\omega_0+\xi)=\mathrm{k}(\omega_0)+k_{\mathrm{mod}}(\xi), \tag{8}$$

where the modified dispersion parameter

$$k_{\mathrm{mod}}(\xi)=\frac{\xi}{c_\infty}+\frac{m''\xi}{2c_\infty(1+i\xi\tau''_{\mathrm{relax}})} \tag{9}$$

depends on new parameters of relaxation

$$m''=\frac{m'}{[1+i\tau_{\mathrm{relax}}\omega_0]^2}, \quad \tau''_{\mathrm{relax}}=\frac{\tau_{\mathrm{relax}}}{[1+i\tau_{\mathrm{relax}}\omega_0]}. \tag{10}$$

Since the integrated function in Eq. (7) decays exponentially as ξ approaches infinity, the Fourier integral Eq. (7) can be represented using the dispersion relation Eq. (8) as

$$\langle p(z,t)\rangle = \frac{e^{i\omega_0 t - ik(\omega_0)z}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi t - ik_{\mathrm{mod}}(\xi)z}\overline{M}(\xi)d\xi - \frac{4e^{i\omega_0 t}}{\pi} \times \int_0^{\pi/2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi t - ik_{\mathrm{mod}}(\xi)(z^2+4a^2\cos^2\theta)^{1/2}} \times \overline{M}(\xi)d\xi\right)e^{-ik(\omega_0)(z+4a^2\cos^2\theta)^{1/2}}\sin^2\theta\, d\theta. \tag{11}$$

The Fourier integrals on the right-hand side of Eq. (11) can be replaced by a convolution integral using the well-known expression for the Green's function as suggested by Vainshtein[11] and Dunin,[12]

$$g_{\mathrm{mod}}(\zeta'',t_{\mathrm{ret}})=\exp(-\zeta'')\,\delta(t_{\mathrm{ret}})+H(t_{\mathrm{ret}}) \times\left(\frac{\zeta''}{t_{\mathrm{ret}}\tau''_{\mathrm{relax}}}\right)^{1/2}\exp\left(-\left|\zeta''+\frac{t_{\mathrm{ret}}}{\tau''_{\mathrm{relax}}}\right|\right) \times I_1[2(\zeta'' t_{\mathrm{ret}}/\tau''_{\mathrm{relax}})^{1/2}], \tag{12}$$

which describes the evolution of a delta-function input in a relaxing medium with modified parameters of relaxation Eq. (10). Here I_1 is the modified Bessel function of the first kind. In addition, t_{ret} is the retarded time and ζ'' is the dimensionless complex number depending on the separation z having the forms

$$t_{\mathrm{ret}}=t-z/c_\infty, \quad \zeta''=m''z/2c_\infty\tau''_{\mathrm{relax}}. \tag{13}$$

Using variables Eq. (13) for brevity, the convolution integral Eq. (11) takes the form

$$\langle p(z,t)\rangle = e^{i\omega_0 t - ik(\omega_0)z}\int_{-\infty}^{+\infty}\left[g_{\mathrm{mod}}(\zeta'',t_{\mathrm{ret}}-s) - \frac{4e^{ik(\omega_0)z}}{\pi} \times\int_0^{\pi/2} g_{\mathrm{mod}}(Z''(\theta),t_{\mathrm{ret}}-s) \times e^{-ik(\omega_0)Z(\theta)}\sin^2\theta\, d\theta\right]M(s)ds, \tag{14}$$

where function $\zeta''(\theta)$ is calculated by means of Eq. (13), s is a dummy parameter of integration, and

$$Z(\theta)=(z^2+4a^2\cos^2\theta)^{1/2}, \quad Z''(\theta)=\left(\zeta''^2+4\left(\frac{2c_\infty\tau''_{\mathrm{relax}}a}{m''}\right)^2\cos^2\theta\right)^{1/2}. \tag{15}$$

Equation (14) is a generalization of the well-known expression for the diffraction correction of harmonic wave in a lossless medium of Williams.[4] However, Eq. (14) permits a burst via the envelope $M(t)$ and includes relaxational effects via the complex parameter Z'', which depends on the relaxation time. It is represented as the product of an oscillatory function and a convolution integral, which can be interpreted as the evolution of a complex envelope function. The integration of the first term in the bracket of the convolution integral describes the evolution of the burst envelope in a one-dimensional relaxing medium, whereas the integration of the second term describes the evolution of its diffraction correction. The attenuation of the harmonic wave portion of the burst signal does not depend on attenuation of the envelope function $M(t)$. However, the evolution of the complex envelope function represented by the integral in Eq. (14) is affected by the frequency of the oscillation of the harmonic wave portion of the signal due to the modified kernel for the relaxing medium with modified parameters related to the net increase in phase speed and the relaxation time Eq. (10).

It is convenient for analysis purposes to transform the convolution integral in Eq. (14) to a dimensionless form. We introduce the dimensionless retarded time normalized by the relaxation time τ_{relax}, and the dimensionless separation between piston and receiver

$$\tau=t_{\mathrm{ret}}/\tau_{\mathrm{relax}}, \quad \overline{\tau}=s/\tau_{\mathrm{relax}}, \quad \zeta=m'z/2c_\infty\tau_{\mathrm{relax}}, \tag{16}$$

so that

$$Z''(\theta)=\left(\zeta^2+4\left(\frac{2c_\infty\tau_{\mathrm{relax}}a}{m'}\right)^2\cos^2\theta\right)^{1/2}. \tag{17}$$

Then the average pressure Eq. (14) can be written as

$$\langle p(z,t)\rangle = e^{i\omega_0 t - ik(\omega_0)z}\int_{-\infty}^{+\infty}\left[G(\zeta,\tau-\overline{\tau},\omega_0,\tau_{\mathrm{relax}}) - \frac{4e^{ik(\omega_0)z}}{\pi}\int_0^{\pi/2} G(Z(\theta),\tau-\overline{\tau},\omega_0,\tau_{\mathrm{relax}}) \times e^{-ik(\omega_0)Z(\theta)}\sin^2\theta\, d\theta\right]M(\overline{\tau})d\overline{\tau}, \tag{18}$$

where the dimensionless Green's function $G(\zeta,\tau,\omega_0,\tau_{\mathrm{relax}})$ takes the form

$$G(\zeta,\tau,\omega_0,\tau_{\text{relax}})=\exp\left(-\frac{\zeta}{1+i\tau_{\text{relax}}\omega_0}\right)\left[\delta(\tau)+H(\tau)\times\left(\frac{\zeta}{\tau}\right)^{1/2}\exp[-\tau(1+i\tau_{\text{relax}}\omega_0)]\times I_1[2(\zeta\tau)^{1/2}]\right]. \quad (19)$$

In Eq. (18), the first term in the large brackets is the plane wave portion, while the second term in the brackets reflects the three-dimensional character of the wave.

The diffraction correction integral term in Eq. (18) for the average pressure depends on the slowly varying envelope function $M(\tau)$ and the plane harmonic wave oscillation ω_0. However, it is possible to asymptotically simplify the diffraction correction in the case of the short wave limit by applying Laplace's method for contour integrals (see Ref. 18, Theorem 6.1, p. 125) to the internal integral over a nonzero separation z, comparable with the radius of the piston. This yields an asymptotic series in powers of $\Omega_0=a\omega_0/c_\infty$, calculated in two stationary points $\theta=0$ and $\theta=\pi/2$. Only the point $\theta=\pi/2$ contributes to the leading term in this asymptotic representation. This is a result of the edge of the circular piston dominating the diffraction correction term. At high frequency, $\Omega_0\gg 0$, the asymptotic representation of the average pressure is

$$\langle p(z,t)\rangle\approx\left(1-e^{-i\pi/4}\sqrt{\frac{2z}{\pi k(\omega_0)a^2}}\right)e^{i\omega_0 t-ik(\omega_0)z}\times\int_{-\infty}^{+\infty}G(\zeta,\tau-\overline{\tau},\omega_0,\tau_{\text{relax}})M(\overline{\tau})d\overline{\tau}. \quad (20)$$

Here, the term in parentheses is the three-dimensional diffraction correction, the factor $\exp(i\omega_0 t-ik(\omega_0)z)$ describes the plane wave evolution of harmonic oscillation including the effect of attenuation due to relaxational losses, and the integral describes the evolution of the envelope $M(t)$ of the burst Eq. (2). The form resembles the appearance of the evolution of a single pulse if the frequency of oscillation $\omega_0=0$ and the product $\tau_{\text{relax}}\omega_0$ in the kernel Eq. (19) is set to zero.[11]

If, in addition, the period of harmonic oscillation $1/\tau_{\text{relax}}\omega_0$ is much less than the envelope duration, Eq. (20) can be simplified to

$$\langle p(z,t)\rangle\approx\left(1-e^{-i\pi/4}\sqrt{\frac{2z}{\pi k(\omega_0)a^2}}\right)e^{i\omega_0 t-\alpha z}M(t). \quad (21)$$

Here $\alpha=m'/2c_\infty\tau_{\text{relax}}$, which does not depend on the frequency of modulation ω_0, is the relaxational attenuation at a frequency well above the frequency associated with the maximum relaxational attenuation. Instead of the envelope $M(t)$ being part of a complex integral as in Eq. (20), the envelope in Eq. (21) appears as the initial envelope. The relaxational attenuation of the modulated pulse depends only on $\exp(i\omega_0 t-az)$ and the diffraction correction in parentheses in Eq. (21). Thus, the result is that the envelope $M(t)$ is attenuated by the factors preceding it, but is not distorted in any way.

Another simplification of Eq. (18) is to consider a plane burst rather than a three-dimensional burst in a relaxing medium. In this case, the second term in the brackets in Eq. (18) drops out. Thus,

$$\langle p(z,t)\rangle=e^{i\omega_0 t-ik(\omega_0)z}\times\int_{-\infty}^{+\infty}G(\zeta,\tau-\overline{\tau},\omega_0,\tau_{\text{relax}})M(\overline{\tau})d\overline{\tau}=e^{i\omega_0 t-ik(\omega_0)z}M_{\text{relax}}(\tau). \quad (22)$$

This is an exact equation, which is a generalization of the solution for a single pulse in a relaxing medium[11–16] that allows for a plane burst rather than a single pulse.

III. ANALYSIS AND NUMERICAL EXAMPLES

We analyze evolution of a planar burst in a relaxing medium using the exact expression Eq. (22), in which relaxation and the burst envelope are factors. The convolution integral in Eq. (22) with kernel Eq. (19) describes evolution of a complex burst envelope, $M_{\text{relax}}(\tau)$, of an oscillating pulse in the same form as the evolution of a single pulse.[16] An important distinction from the case of single pulse propagation is the oscillatory nature of the complex kernel as the function of retarded time, depending on the relaxation time and the frequency of oscillation. The interplay of these independent parameters results in an anomalous behavior of the complex envelope, which is quite different from that for a single pulse.

To demonstrate the interplay between the burst frequency and the burst envelope, we calculate the absolute value of the burst envelope for different separations between the piston and the receiver for a variety of oscillation frequencies using a Gaussian envelope $M(t)=\exp(-t^2/2\gamma^2\tau_{\text{relax}}^2)$. The following parameters were used for the calculation: radius of piston $a=0.01$ m, the frozen speed of sound $c_\infty=300$ m/s, relaxation time $\tau_{\text{relax}}=10^{-7}$ s, and the net increase in phase speed $m'=10^{-3}$. These parameters are typical of ultrasonic transducers in air. The parameter γ defines the width of the envelope. The dimensionless separation ζ between the piston and the receiver is defined by Eq. (16).

It is instructive to investigate the evolution of the envelope using Eq. (22) and varying the duration of the Gaussian envelope and dimensionless frequency $\omega_0\tau_{\text{relax}}$. First let the parameter $\gamma=1$ in the Gaussian envelope and let the dimensionless frequency of the oscillations in the burst be $0\leq\omega_0\tau_{\text{relax}}\leq 20$. In Fig. 1(a), the shape of the transmitted burst and its envelope are shown for $\omega_0\tau_{\text{relax}}=0,0.5,1,2,3,3.5,10,20$, corresponding to 0 to 20 oscillations within the burst envelope. Figure 1(b) shows the evolution of the burst envelope, $M_{\text{relax}}(\tau)$, which corresponds to the integral in Eq. (22). Here the horizontal axis is the dimensionless retarded time $\tau=t_{\text{ret}}/\tau_{\text{relax}}$, and envelope profiles are the calculated at dimensionless distances from the piston of $\zeta=0,2,4,6,8$. The curve for $\zeta=0$ is the Gaussian

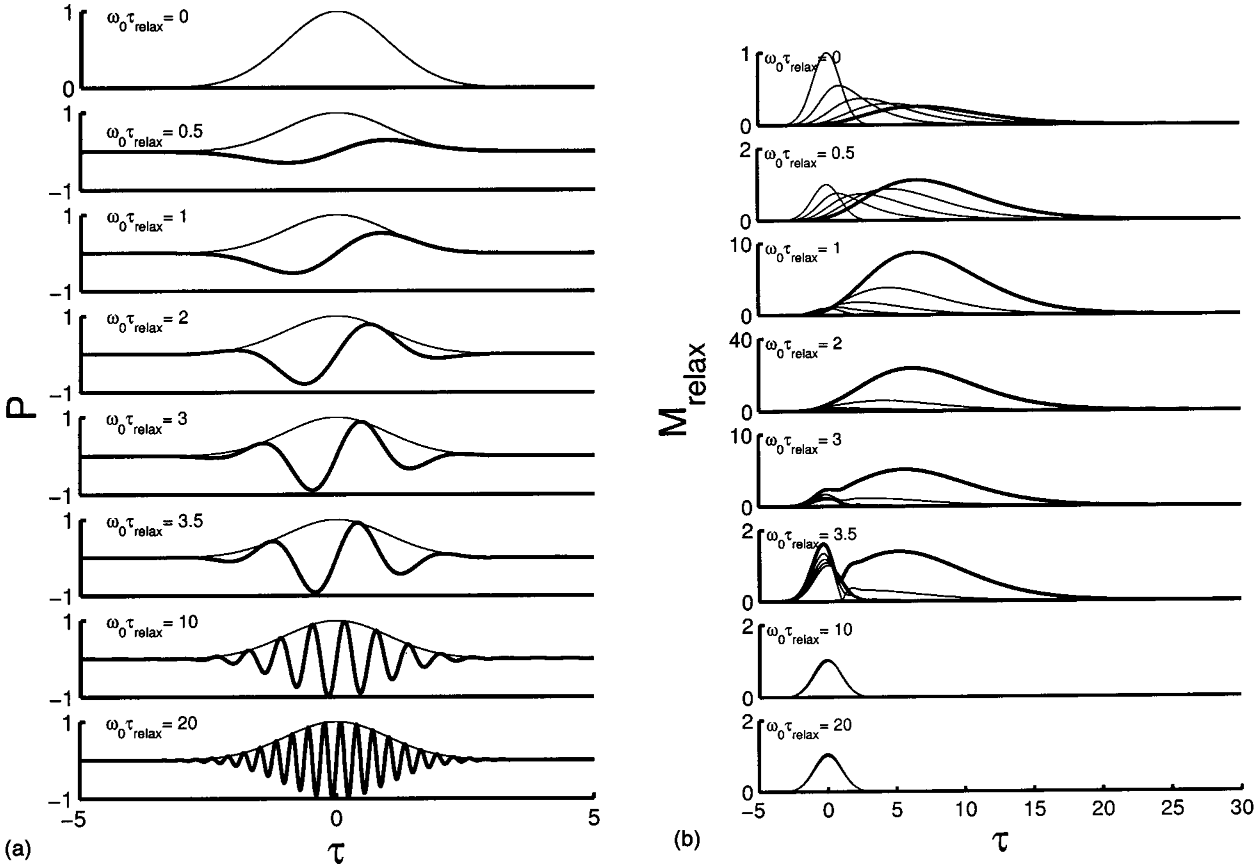

FIG. 1. (a) Gaussian bursts for $\gamma=1$ for different values of the dimensionless parameter $\omega_0\tau_{\text{relax}}=0,0.5,1,2,3,3.5,10,20$. (b) Evolution of the burst envelopes for $\omega_0\tau_{\text{relax}}=0,0.5,1,2,3,3.5,10,20$ at dimensionless separations of $\zeta=0,2,4,6,8$. The Gaussian curve at $\tau=0$ corresponds to $\zeta=0$; the bold curve corresponds to $\zeta=8$ for all cases except $\omega_0\tau_{\text{relax}}=10$, 20 where all curves overlap.

curve centered at $\tau=0$; the curve for $\zeta=8$ is bold; curves for $\zeta=2,4,6$ lie between the $\zeta=0$ and $\zeta=8$ curves.

For $\omega_0\tau_{\text{relax}}=0$, which is the case of a single pulse, the evolution of the complex envelope degenerates to the evolution of the Gaussian single pulse. The pulse amplitude decreases with distance, while the pulse duration increases. Andreev *et al.*[16] observed a similar broadening of a single pulse and a decrease in the pulse amplitude.

For $\omega_0\tau_{\text{relax}}=0.5$, similar envelope broadening occurs, accompanied by decreasing amplitude for short distances from the piston. However, for $\zeta\geqslant 6$ the maximum of the burst envelope increases with distance. Such amplification causes the attenuation of the oscillating pulse to differ substantially from the attenuation of a harmonic wave or a single pulse. The combined effects of the burst oscillation, the burst envelope, and the relaxation of the media produce this result. This amplification can be quite large, noting the different vertical scales used in Fig. 1(b). The amplification reaches a maximum for the case of $\omega_0\tau_{\text{relax}}=2$, where the maximum amplitude of the burst envelope is over a magnitude larger than that of the transmitted burst. For $3\leqslant\omega_0\tau_{\text{relax}}\leqslant 3.5$, the amplification weakens and the envelope splits into in two humps. The weakening may be attributed to the oscillation of the second term in the kernel Eq. (19) as a function of τ, which leads to degradation of its convolution with the burst envelope and eventual domination of the convolution with the first term. As the oscillation frequency increases further, the amplification decreases, so that for $\omega_0\tau_{\text{relax}}=10$, 20 the amplification is negligible and the burst duration is unchanged because the oscillating pulse attenuates as a plane harmonic wave.

The maxima of the envelope magnitudes are shown in Fig. 2 as functions of the dimensionless parameter $\omega_0\tau_{\text{relax}}$ for distances from the piston of $\zeta=0,2,4,6,8$. The horizontal

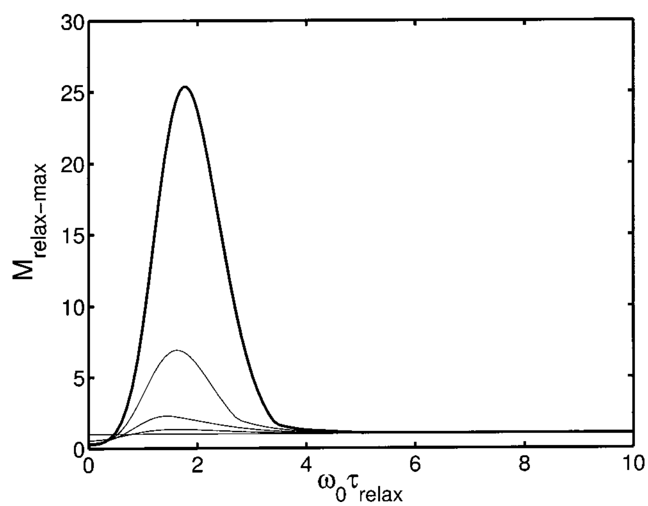

FIG. 2. Maximum values of the Gaussian burst envelopes calculated at dimensional separations of $\zeta=0,2,4,6,8$ for $\gamma=1$. The horizontal line at $M_{\text{relax\_max}}=1$ corresponds to $\zeta=0$; the bold curve corresponds to $\zeta=8$.

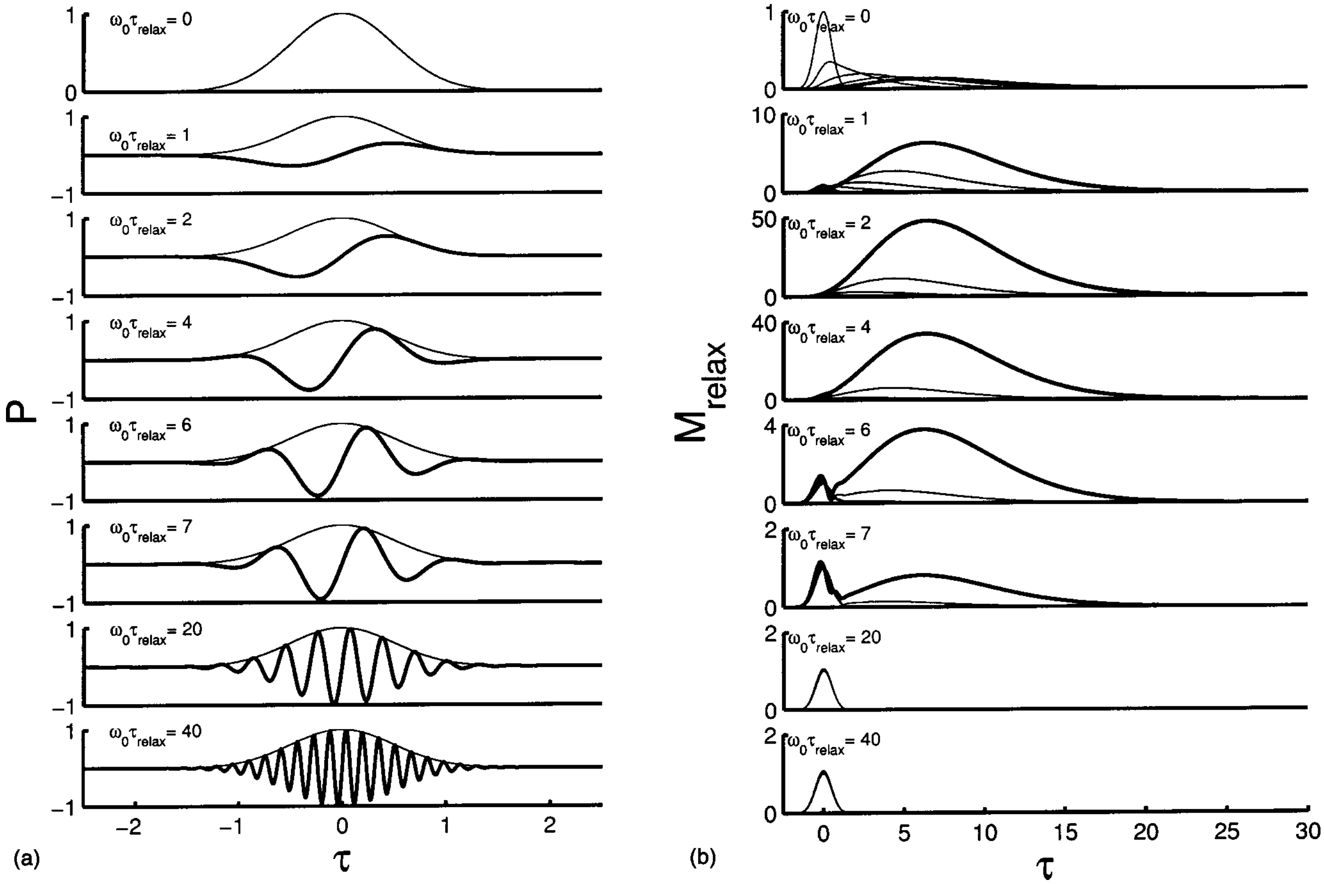

FIG. 3. (a) Gaussian bursts for $\gamma=0.5$ for different values of the dimensionless parameter $\omega_0\tau_{\mathrm{relax}}=0,1,2,4,6,7,20,40$. (b) Evolution of the burst envelopes for $\omega_0\tau_{\mathrm{relax}}=0,1,2,4,6,7,20,40$ at dimensionless separations of $\zeta=0,2,4,6,8$. The Gaussian curve at $\tau=0$ corresponds to $\zeta=0$; the bold curve corresponds to $\zeta=8$ for all cases except $\omega_0\tau_{\mathrm{relax}}=20$, 40 where all curves overlap.

line at $M_{\mathrm{relax\_max}}=1$ corresponds to $\zeta=0$; the bold curve corresponds to $\zeta=8$. The magnitude of the envelope grows quite large as the separation distance ζ increases for $1\leqslant\omega_0\tau_{\mathrm{relax}}\leqslant 3$.

Now we compare this result for $\gamma=1$ with the evolution of the Gaussian envelope for shorter and longer durations, $\gamma=0.5$ and $\gamma=2$. In Fig. 3(a) for $\gamma=0.5$, half as many oscillations are included in the shorter burst envelope than for the duration $\gamma=1$ at the same $\omega_0\tau_{\mathrm{relax}}$. In Fig. 3(b), the amplification of the envelope occurs over a wider range of the dimensionless frequency, $1\leqslant\omega_0\tau_{\mathrm{relax}}\leqslant 6$. In addition, the maximum amplitudes are higher than those for a wider envelope, $\gamma=1$ in Fig. 1(b). This is also reflected in Fig. 4, when compared to Fig. 2. The maximum amplification can be more than twice as much for the shorter envelope, noting the differences in the vertical scales in the figures.

For $\gamma=2$, more oscillations are included within the longer envelope than for $\gamma=1$ at the same $\omega_0\tau_{\mathrm{relax}}$ as shown in Fig. 5(a), noting the different horizontal scale from previous similar figures. Like the previous two cases, amplification of the envelope occurs as shown in Fig. 5(b), but in this case the range of frequencies is smaller, $0.5\leqslant\omega_0\tau_{\mathrm{relax}}\leqslant 2$. In addition, the maximum amplitudes are much smaller. These results are more evident comparing Fig. 6 to Figs. 2 and 4, noting the substantially different vertical scales.

In spite of the differences in the three cases, the amplification always occurs when $\omega_0\tau_{\mathrm{relax}}$ has an order of magnitude of unity. However, the degree of amplification and the frequency at which the maximum occurs both increase as the envelope shortens. It appears that when the period of the pulse is similar to the duration of the envelope, the amplification occurs. For example, the combinations ($\omega_0\tau_{\mathrm{relax}}=4$, $\gamma=0.5$), ($\omega_0\tau_{\mathrm{relax}}=2$, $\gamma=1$), and ($\omega_0\tau_{\mathrm{relax}}=1$, $\gamma=2$) all consist of one full wave within the burst envelope [see Figs.

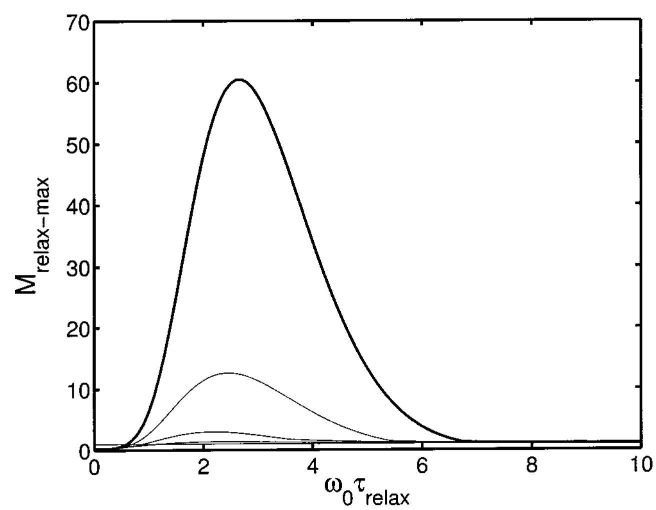

FIG. 4. Maximum values of the Gaussian burst envelopes calculated at dimensional separations of $\zeta=0,2,4,6,8$ for $\gamma=0.5$. The horizontal line at $M_{\mathrm{relax\_max}}=1$ corresponds to $\zeta=0$; the bold curve corresponds to $\zeta=8$.

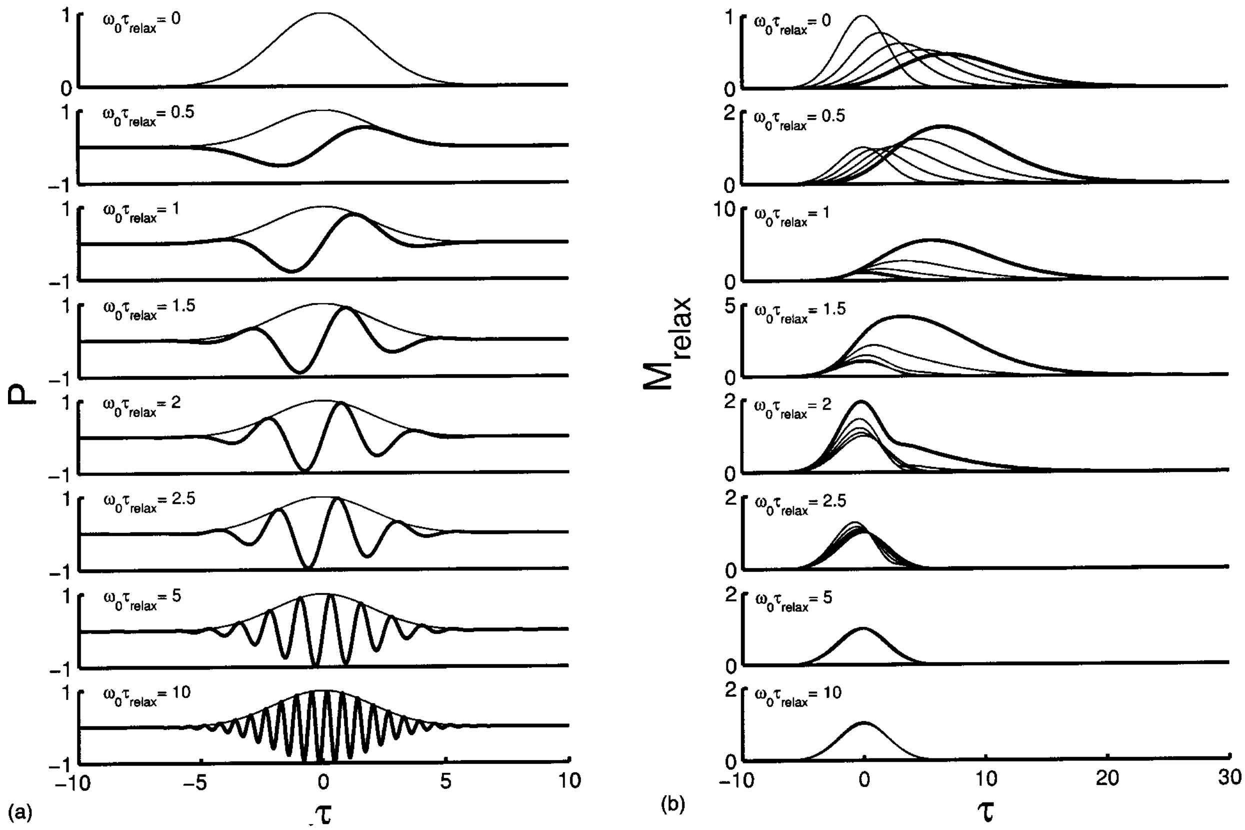

FIG. 5. (a) Gaussian bursts for $\gamma=2$ for different values of the dimensionless parameter $\omega_0\tau_{relax}=0,0.5,1,1.5,2,2.5,5,10$. (b) Evolution of the burst envelopes for $\omega_0\tau_{relax}=0,0.5,1,1.5,2,2.5,5,10$ at dimensionless separations of $\zeta=0,2,4,6,8$. The Gaussian curve at $\tau=0$ corresponds to $\zeta=0$; the bold curve corresponds to $\zeta=8$ for all cases except $\omega_0\tau_{relax}5,10$ where all curves overlap.

1(a), 3(a), and 5(a)]. In all three cases, the maximum amplification occurs very near these frequencies, as shown in Figs. 2, 4, and 6.

The nature of the amplification of the burst envelope can be understood by looking into dependence of the kernel Eq. (19) on the dimensionless frequency $\omega_0\tau_{relax}$. The kernel, G, is a product of two factors: the first one, $\exp(-\zeta/(1+i\omega_0\tau_{relax}))$, depends on the distance from the piston; the second factor, which is in brackets in Eq. (19), is the convolution with the Gaussian envelope. Its second term oscillates as $\exp(-i\omega_0\tau_{relax})$. When the period of Gaussian envelope is comparable with the period of oscillation, the second term in the brackets dominates, and the convolution of the oscillating term with $M(\tau)$ is an increasing function of distance ζ. Consequently, the maximum amplitude of the envelope increases with distance. However, the rate of the growth of the envelope's amplitude is modulated by the factor in the kernal preceding the bracketed term, $\exp(-\zeta/(1+i\omega_0\tau_{relax}))$. Of course, for larger $\omega_0\tau_{relax}$, this term is less significant. Thus, for small γ, where amplification occurs for a larger value of $\omega_0\tau_{relax}$, the envelope is amplified more than for large γ, where the amplification occurs at a smaller value of $\omega_0\tau_{relax}$.

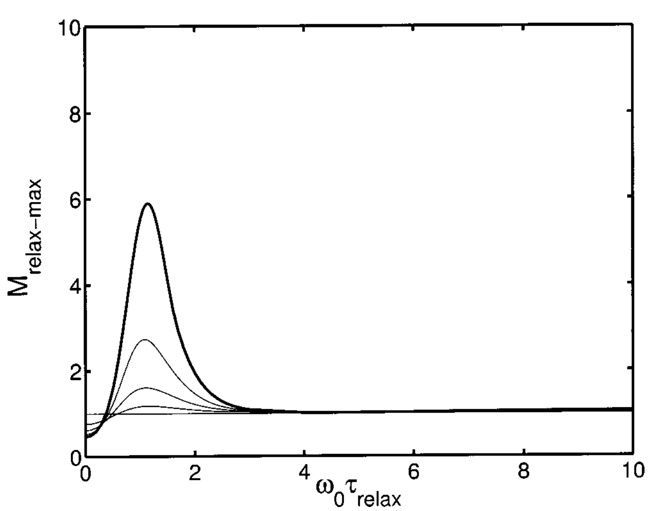

FIG. 6. Maximum values of the Gaussian burst envelopes calculated at dimensional separations of $\zeta=0,2,4,6,8$ for $\gamma=2$. The horizontal line at $M_{relax\_max}=1$ corresponds to $\zeta=0$; the bold curve corresponds to $\zeta=8$.

From Figs. 2, 4, and 6 there appears to be frequency at which there is a change from amplification to a situation of nearly no amplification, which is characteristic of a harmonic wave. This occurs when $\omega_0\tau_{relax}$ is sufficiently high so that the oscillation of the second term in the kernel yields a convolution integral with very small magnitude, and the exponential factor $\exp(-\zeta/(1+i\omega_0\tau_{relax}))$ dominates the evolution of the complex envelope.

IV. CONCLUDING REMARKS

Equation (18) describes the interaction between the diffractional attenuation and the relaxational attenuation as a tone burst propagates between circular transducers in a relaxing medium. Applying this formulation to the more simple case of a planar burst demonstrates the interaction between the burst duration, the harmonic frequency, and the relaxation time constant. When the duration of the burst coincides with the relaxation time and the period of the wave, amplification of the burst envelope occurs. When the duration of the burst is substantially longer than the period of the wave the attenuation is similar to that for a harmonic wave. Provided that the relaxation time is known, Eq. (18) provides a means for determining the combined effect of diffraction and relaxation on the attenuation of a tone burst.

ACKNOWLEDGMENTS

This work was supported by DOE under subcontract to Commercial Electronics, Inc., Broken Arrow, OK, and by the Ford Motor Company. We thank Dr. Jaco Visser and Scott Phillips for several helpful discussions.

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